(56)複數

請看以下的方程式:

$x^2 + \mathrm{x} + 3 = 0$

我們可以用公式來求答案:

$$\mathrm{x} = \frac{-1 \pm \sqrt{1^2 - 4 \times 1 \times 3}}{2} = \frac{-1 \pm \sqrt{1 - 12}}{2} = \frac{-1 \pm \sqrt{-11}}{2} = -\frac{1}{2} \pm \frac{\sqrt{11}}{2} i$$

因此，我們得到了一個$-\frac{1}{2} \pm \frac{\sqrt{11}}{2} i$的數。這個數比純的虛數要複雜一點，因為它有實數部分。

如果 a 與 b 都是實數，則$\mathrm{a} + \mathrm{b}i$是一個複數。因此，以下的數都是複數:

$1 + 2i$
$-3 + 5i$
$-\frac{1}{2} - \frac{2}{3} i$
$5 + \sqrt{6} i$
$\sqrt{7} - \sqrt{6} i$

複數的加減乘除

(1)
(a)$(3 + 5i) + (4 - 6i) = (3 + 4) + (5i - 6i) = 7 + (-i) = 7 - i$

(b)$-(2 + 3i) + (5 - 7i) = -2 - 3i + 5 - 7i = (-2 + 5) + (-3i - 7i) = 3 + (-10i) = 3 - 10i$

(c) $(5 - 3i) - (-7 + 9i) = 5 - (-7) - 3i - 9i = 12 - 12i$

(d) $(8 + 3i) - \left(\sqrt{2} - \sqrt{3} i\right) = \left(8 - \sqrt{2}\right) + \left(3i + \sqrt{3} i\right) = \left(8 - \sqrt{2}\right) + \left(3 + \sqrt{3}\right) i$

(2)
(a)$(3 + 5i)(4 - 6i) = (3 \times 4) + 3(-6i) + (5i) \times 4 + (5i) \times (-6i) = 12 - 18i +$

$20i-30i^2=12+2i-30(-1)=42+2i$

(b)$(2+3i)(5-7i)=(2\times5)+2(-7i)+(3i)5+(3i)(-7i)=10-14i+15i-21i^2=10+i-21(-1)=31+i$

(c) $(3+i)(3-i)=9-i^2=9+1=10$

(d) $(3i+1)(3i-1)=9i^2-1=-9-1=-10$

(e)$(i+1)^2=i^2+2i+1=-1+2i+1=2i$

(f) $(3i-2)^2=9i^2-12i+4=-9-12i+4=-5-12i$

(g) $(-i+2)^2=(-i)^2-4i+4=i^2-4i+4=3-4i$

(h) $\left(2+\sqrt{2}i\right)\left(\sqrt{8}i+3\right)=2\sqrt{8}i+6+\sqrt{2}\sqrt{8}i^2+3\sqrt{2}i=6+\sqrt{2}\sqrt{8}(-1)+2\sqrt{8}i+3\sqrt{2}i=\left(6-\sqrt{16}\right)+2\times2\sqrt{2}i+3\sqrt{2}i=(6-4)+7\sqrt{2}i=2+7\sqrt{2}i$

(3)除法

(a)$\frac{3-i}{1-i}=\frac{(3-i)(1+i)}{(1-i)(1+i)}=\frac{3+3i-i-i^2}{1-i^2}=\frac{3-(-1)+2i}{1-(-1)}=\frac{4+2i}{2}=2+i$

(b) $\frac{1}{1-i}=\frac{(1+i)}{(1-i)(1+i)}=\frac{1+i}{1-i^2}=\frac{1+i}{1-(-1)}=\frac{1+i}{2}=\frac{1}{2}+\frac{i}{2}$

(c) $\frac{4+i}{3-i}=\frac{(4+i)(3+i)}{(3-i)(3+i)}=\frac{12+4i+3i+i^2}{9-i^2}=\frac{12-1+7i}{9-(-1)}=\frac{11+7i}{10}=\frac{11}{10}+\frac{7}{10}i$

(d)$\frac{1}{(3+i)^2}=\frac{1}{9+6i+i^2}=\frac{1}{9-1+6i}=\frac{1}{8+6i}=\frac{8-6i}{(8+6i)(8-6i)}=\frac{8-6i}{64-(6i)^2}=\frac{8-6i}{64+36}=\frac{8-6i}{100}=\frac{8}{100}-\frac{6}{100}i=\frac{2}{25}-\frac{3}{50}i$

(e) $\frac{i+1}{3i-5}=\frac{(i+1)(3i+5)}{(3i-5)(3i+5)}=\frac{3i^2+5i+3i+5}{(3i)^2-25}=\frac{-3+5+8i}{-9-25}=\frac{2+8i}{-34}=-\frac{1}{17}-\frac{4}{17}i$

(f) $\frac{1}{(i+2)^2}=\frac{1}{i^2+4i+4}=\frac{1}{-1+4i+4}=\frac{1}{3+4i}=\frac{3-4i}{(3+4l)(3-4i)}=\frac{3-4i}{9-(4i)^2}=\frac{3-4i}{9+16}=\frac{3-4i}{25}=\frac{3}{25}-$

$\frac{4}{25}i$

(g) $\frac{2+i}{3-5i}=\frac{(2+i)(3+5i)}{(3-5i)(3+5i)}=\frac{6+10i+3i+5i^2}{9-(5i)^2}=\frac{6-5+13i}{9+25}=\frac{1+13i}{34}=\frac{1}{34}+\frac{13}{34}i$

(4)一元二次方程式

(a)$x^2+x+1=0$

$$x=\frac{-1\pm\sqrt{1^2-4}}{2}=\frac{-1}{2}\pm\frac{\sqrt{3}}{2}i$$

我們可以以$x=\frac{-1}{2}+\frac{\sqrt{3}}{2}i$查看答案是否正確

x^2+x+1

$$=\left(\frac{-1}{2}+\frac{\sqrt{3}}{2}i\right)^2+\left(\frac{-1}{2}+\frac{\sqrt{3}}{2}i\right)+1$$

$$=\frac{1}{4}-\frac{\sqrt{3}}{2}i+\frac{3}{4}i^2-\frac{1}{2}+\frac{\sqrt{3}}{2}i+1$$

$$=\frac{1}{4}-\frac{3}{4}-\frac{1}{2}+1=0$$

可見得$x=\frac{-1}{2}+\frac{\sqrt{3}}{2}i$是正確的

(b) $x^2-2x+2=0$

$$x=\frac{2\pm\sqrt{(-2)^2-4\times 2}}{2}=\frac{2\pm\sqrt{-4}}{2}=\frac{2\pm 2i}{2}=1\pm i$$

我們以$x=1-i$來看答案是否正確

x^2-2x+2

$=(1-i)^2-2(1-i)+2$

$=1-2i+i^2-2+2i+2$

$= 1 + i^2 = 0$

(c) $x^2 - x + 2 = 0$

$$x = \frac{1 \pm \sqrt{(-1)^2 - 4 \times 2}}{2} = \frac{1 \pm \sqrt{-7}}{2} = \frac{1}{2} \pm \frac{\sqrt{7}}{2} i$$

我們以$x = \frac{1}{2} + \frac{\sqrt{7}}{2} i$來看答案是否正確

$x^2 - x + 2$

$$= \left(\frac{1}{2} + \frac{\sqrt{7}}{2} i\right)^2 - \left(\frac{1}{2} + \frac{\sqrt{7}}{2} i\right) + 2$$

$$= \frac{1}{4} + \frac{\sqrt{7}}{2} i + \frac{7}{4} i^2 - \frac{1}{2} - \frac{\sqrt{7}}{2} i + 2$$

$$= \frac{1}{4} - \frac{7}{4} - \frac{1}{2} + 2 = 0$$

(d) $2x^2 - x + 1 = 0$

$$x = \frac{1 \pm \sqrt{(-1)^2 - 4 \times 2 \times 1}}{2 \times 2} = \frac{1 \pm \sqrt{-7}}{4} = \frac{1}{4} \pm \frac{\sqrt{7}}{4} i$$

我們以$x = \frac{1}{4} + \frac{\sqrt{7}}{4} i$來看答案是否正確

$2x^2 - x + 1$

$$= 2\left(\frac{1}{4} + \frac{\sqrt{7}}{4} i\right)^2 - \left(\frac{1}{4} + \frac{\sqrt{7}}{4} i\right) + 1$$

$$= 2\left(\frac{1}{16} + \frac{\sqrt{7}}{8} i + \frac{7}{16} i^2\right) - \frac{1}{4} - \frac{\sqrt{7}}{4} i + 1$$

$$= \frac{1}{8} + \frac{\sqrt{7}}{4} i - \frac{7}{8} - \frac{1}{4} - \frac{\sqrt{7}}{4} i + 1$$

$$= \frac{1}{8} - \frac{7}{8} - \frac{1}{4} + 1 = 0$$

(5)帶虛數的方程式

(a)$2Z + 3i = 5$

$2Z = 5 - 3i$

$Z = \frac{5}{2} - \frac{3}{2}i$

(b) $2Z - iZ = 5$

$Z(2 - i) = 5$

$Z = \frac{5}{2-i} = \frac{5(2+i)}{(2-i)(2+i)} = \frac{5(2+i)}{4-i^2} = \frac{5(2+i)}{5} = 2 + i$

我們可以將$Z = 2 + i$代入原方程式，看看答案對不對

$2Z - iZ = 5$

左式=$2(2 + i) - i(2 + i)$

$= 4 + 2i - 2i - i^2$

$= 4 - (-1) = 5$=右式

我們可知答案是對的

(c)$(3 + i)Z = 5iZ - 3$

$(3 + i - 5i)Z = -3$

$(3 - 4i)Z = -3$

$Z = \frac{-3}{3-4i} = \frac{-3(3+4i)}{(3-4i)(3+4i)} = \frac{-9-12i}{9-(-16)} = \frac{-9}{25} - \frac{12}{25}i$

將此答案代入原方程式

左式=$(3 + i)\left(\frac{-9}{25} - \frac{12}{25}i\right)$

$= \frac{-27}{25} - \frac{36}{25}i - \frac{9}{25}i - \frac{12}{25}i^2$

$= \frac{-27}{25} + \frac{12}{25} - \frac{45}{25}i$

$= \frac{-15}{25} - \frac{45}{25}i = \frac{-3}{5} - \frac{9}{5}i$

右式=$5i\left(\frac{-9}{25} - \frac{12}{25}i\right) - 3$

$= \frac{-9}{5}i - \frac{12}{5}i^2 - 3 = \frac{-9}{5}i - \frac{3}{5}$

左式=右式，答案正確。

(d) $(1+i)\mathrm{Z} = 2 - \mathrm{Z}$

$(1+i+1)\mathrm{Z} = 2$

$(2+i)\mathrm{Z} = 2$

$\mathrm{Z} = \frac{2}{2+i} = \frac{2(2-i)}{(2+i)(2-i)} = \frac{4-2i}{4-(-1)} = \frac{4}{5} - \frac{2}{5}i$

查看答案

左式=$(1+i)\mathrm{Z}$

$= (1+i)\left(\frac{4}{5} - \frac{2}{5}i\right)$

$= \frac{4}{5} - \frac{2}{5}i + \frac{4}{5}i - \frac{2}{5}i^2$

$= \frac{4}{5} + \frac{2}{5} + \frac{2}{5}i = \frac{6}{5} + \frac{2}{5}i$

右式=$2 - \mathrm{Z}$

$= 2 - \left(\frac{4}{5} - \frac{2}{5}i\right)$

$= 2 - \frac{4}{5} + \frac{2}{5}i = \frac{6}{5} + \frac{2}{5}i$

左式=右式，答案正確。