

(41) 等比級數

等比級數的數列定義如下：

$$a_1, a_2, a_3 \cdots a_n$$

$$a, ar, ar^2 \cdots ar^{n-1}$$

因此，在一個等比級數內，

$$a_i = a_{i-1}r \cdots \cdots \cdots (1)$$

$$\frac{a_{i+1}}{a_i} = r \cdots \cdots \cdots (2)$$

$$a_i = ar^{i-1} \cdots \cdots \cdots (3)$$

以下都是等比級數的例子

(1)  $a=2, r=2$

$$2, 4, 8, 16, 32, 64$$

(2)  $a = 1, r = \frac{1}{2}$

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}$$

(3)  $a=1, r=-2$

$$1, -2, 4, -8, 16, -32, 64$$

(4)  $a = 3, r = \frac{1}{3}$

$$3, 1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \frac{1}{243}$$

(5)  $a=2, r=3$ ，第 5 項是什麼？

$$a_5 = ar^{5-1} = 2(3)^4 = 2 \times 81 = 162$$

(6)  $a = 1$ ,  $r = -\frac{1}{2}$ , 第 5 項是什麼?

$$a_5 = ar^{5-1} = 1 \times \left(-\frac{1}{2}\right)^4 = 1 \times \frac{1}{16} = \frac{1}{16}$$

(7)  $a = 1$ ,  $r = -\frac{1}{2}$ , 第 6 項是什麼?

$$a_6 = ar^{6-1} = 1 \times \left(-\frac{1}{2}\right)^5 = -\frac{1}{32}$$

(8)  $\sqrt{2} - 1$ ,  $1$ ,  $a$ ,  $b$  是一等比級數, 求  $r$ ,  $a$  和  $b$

$$r = \frac{a_{i+1}}{a_i} = \frac{a_2}{a_1} = \frac{1}{\sqrt{2}-1} = \frac{\sqrt{2}+1}{(\sqrt{2}-1)(\sqrt{2}+1)} = \sqrt{2}+1$$

$$a = 1 \times r = \sqrt{2}+1$$

$$b = ar = (\sqrt{2}+1)^2$$

以下是這個等比級數

$$\sqrt{2}-1, 1, \sqrt{2}+1, (\sqrt{2}+1)^2$$

(9)  $\frac{1}{\sqrt{5}-1}$ ,  $\frac{1}{4}$ ,  $a$ ,  $b$  是一等比級數, 求  $r$ ,  $a$  和  $b$

$$r = \frac{a_{i+1}}{a_i} = \frac{a_2}{a_1} = \frac{\frac{1}{4}}{\frac{1}{\sqrt{5}-1}} = \frac{\sqrt{5}-1}{4}$$

$$\therefore a = \frac{1}{4} \times \frac{\sqrt{5}-1}{4} = \frac{\sqrt{5}-1}{16}$$

$$b = ar = \frac{\sqrt{5}-1}{16} \times \frac{\sqrt{5}-1}{4} = \frac{(\sqrt{5}-1)^2}{64} = \frac{6-2\sqrt{5}}{64}$$

等比級數的和

$$\text{令 } S_n = a + ar + ar^2 + \dots + ar^{n-1}$$

$$rS_n = ar + ar^2 + \dots + ar^n$$

$$\therefore S_n - rS_n = a - ar^n$$

$$S_n(1 - r) = a(1 - r^n)$$

$$S_n = \frac{a(1 - r^n)}{1 - r} \dots\dots\dots (4)$$

$$(10) \quad a=1, r=2, n=5$$

$$S_n = 1 + 2 + 4 + 8 + 16 = 31$$

用公式(4)

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{1(1 - 2^5)}{1 - 2} = \frac{1(1 - 32)}{-1} = \frac{-31}{-1} = 31$$

$$(11) \quad a = 2, r = \frac{1}{2}, n = 6$$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{2(1 - (\frac{1}{2})^6)}{1 - \frac{1}{2}} = \frac{2(1 - \frac{1}{64})}{\frac{1}{2}} = 4(1 - \frac{1}{64}) = 4(\frac{63}{64}) = \frac{63}{16}$$

同學可以自行驗證答案的正確性