(20) 三角函數的倍角公式

我們首先要導出

$$\sin 2\theta = 2\sin\theta\cos\theta$$

我們可以用

$$\sin 2\theta = \sin(\theta+\theta)$$

$$= \sin\theta\cos\theta + \cos\theta\sin\theta$$

$$= 2\sin\theta\cos\theta$$

我們再證

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$

我們可用

$$\cos 2\theta = \cos(\theta+\theta)$$

$$= \cos\theta\cos\theta - \sin\theta\sin\theta$$

$$= \cos^2\theta - \sin^2\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$= \cos^2\theta - (1-\cos^2\theta)$$

$$= 2\cos^2\theta - 1$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$= 1 - \sin^2\theta - \sin^2\theta$$

$$= 1 - 2\sin^2\theta$$

整理如下:

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\cos 2\theta = 2\cos^2\theta - 1$$

$$\cos 2\theta = 1 - 2\sin^2\theta$$

(1)已知$\sin 30^\circ = \frac{1}{2}$，求$\sin 60^\circ$。

$$\cos 30^\circ = \sqrt{1-\sin^2 30^\circ}$$
$$= \sqrt{1-\frac{1}{4}}$$
$$=\frac{\sqrt{3}}{2}$$
$$\therefore \sin 60^\circ = 2\sin 30^\circ \cos 30^\circ$$
$$= 2\times\frac{1}{2}\times\frac{\sqrt{3}}{2}$$
$$=\frac{\sqrt{3}}{2}$$

(2)已知$\sin 30^\circ = \frac{1}{2}$，求$\cos 60^\circ$。

$$\cos 60^\circ = 1-2\sin^2 30^\circ$$
$$= 1-2\times(\frac{1}{2})^2$$
$$=1-2\times\frac{1}{4}$$
$$=1-\frac{1}{2}$$
$$=\frac{1}{2}$$

(3)證明$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$$\sin 3\theta = \sin(2\theta+\theta)$$
$$= \sin 2\theta\cos\theta + \cos 2\theta\sin\theta$$
$$= 2\sin\theta\cos^2\theta + (1-2\sin^2\theta)\sin\theta$$
$$= 2\sin\theta(1-\sin^2\theta) + \sin\theta - 2\sin^3\theta$$
$$= 3\sin\theta - 4\sin^3\theta$$

(4)證明$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$$
\begin{aligned}
\cos 3\theta &= \cos(2\theta + \theta) \\
&= \cos 2\theta \cos\theta - \sin 2\theta \sin\theta \\
&= (2\cos^2\theta - 1)\cos\theta - 2\sin^2\theta\cos\theta \\
&= 2\cos^3\theta - \cos\theta - 2(1 - \cos^2\theta)\cos\theta \\
&= 4\cos^3\theta - 3\cos\theta
\end{aligned}
$$

(5)證明$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

$$
\begin{aligned}
\tan 2\theta &= \frac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta} \\
&= \frac{\frac{2\sin\theta\cos\theta}{\cos^2\theta}}{1 - \frac{\sin^2\theta}{\cos^2\theta}} \\
&= \frac{2\tan\theta}{1 - \tan^2\theta}
\end{aligned}
$$

(6)證明$\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}$

$$
\begin{aligned}
\sin 2\theta &= 2\sin\theta\cos\theta \\
&= \frac{2\sin\theta\cos\theta}{\cos^2\theta} \times \cos^2\theta \\
&= 2\tan\theta\cos^2\theta \\
&= \frac{2\tan\theta}{\sec^2\theta} \\
&= \frac{2\tan\theta}{1 + \tan^2\theta}
\end{aligned}
$$

(7)證明$cos\,2\,\theta=\frac{1-tan^2\theta}{1+tan^2\theta}$

$cos\,2\,\theta=cos^2\theta-sin^2\theta$

$$=\frac{cos^2\theta-sin^2\theta}{cos^2\theta}\times cos^2\theta$$
$$=\frac{1-tan^2\theta}{sec^2\theta}$$
$$=\frac{1-tan^2\theta}{1+tan^2\theta}$$

(8)已知$sin\,\alpha=\frac{1}{\sqrt{2}}$，$cos\,\beta=\frac{\sqrt{3}}{2}$，求$sin(\alpha+\beta)$。

$sin\,\alpha=\frac{1}{\sqrt{2}}$，$cos\,\alpha=\sqrt{1-\frac{1}{2}}=\frac{1}{\sqrt{2}}$

$cos\,\beta=\frac{\sqrt{3}}{2}$，$sin\,\beta=\sqrt{1-\frac{3}{4}}=\frac{1}{2}$

$\therefore sin(\alpha+\beta)=sin\,\alpha\,cos\,\beta+cos\,\alpha\,sin\,\beta$

$$=\frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{2}}\times\frac{1}{2}$$
$$=\frac{\sqrt{6}}{4}+\frac{\sqrt{2}}{4}$$
$$=\frac{\sqrt{6}+\sqrt{2}}{4}$$

(9)若$sin\,\alpha+sin\,\beta=1$，及$cos\,\alpha+cos\,\beta=0$，求$cos\,2\,\alpha$及$cos\,2\,\beta$。

$\because sin\,\alpha+sin\,\beta=1$

$sin\,\alpha=1-sin\,\beta\cdots\cdots(1)$

$\because cos\,\alpha+cos\,\beta=0$

$\therefore cos\,\alpha=-cos\,\beta$

$(1)^2+(2)^2\rightarrow$

$sin^2\alpha+\cos^2\alpha=(1-sin\,\beta)^2+\cos^2\beta$

$$=1-2\,sin\,\beta+sin^2\beta+1-sin^2\beta$$

$=2-2\sin\beta$

$\therefore 1=2-2\sin\beta$

$\sin\beta=\frac{1}{2}\cdots\cdots(3)$

$\cos 2\beta=1-2\sin^2\beta$

$=1-2\times(\frac{1}{2})^2$

$=1-2\times\frac{1}{4}$

$=\frac{1}{2}\cdots\cdots(4)$

$\because \sin\alpha+\sin\beta=1$

$\therefore \sin\alpha=1-\sin\beta=1-\frac{1}{2}=\frac{1}{2}$

$\cos 2\alpha=1-2\sin^2\alpha$

$=1-2\times(\frac{1}{2})^2$

$=1-2\times\frac{1}{4}$

$=\frac{1}{2}\cdots\cdots(5)$

答案$\cos 2\alpha=\frac{1}{2}$、$\cos 2\beta=\frac{1}{2}$

我們已經學會了很多的倍角公式

現在我們可以驗證一下

(10)$\theta=0^\circ$

$\sin 0^\circ=0$，$\cos 0^\circ=1$

$\sin 2\theta=\sin 0^\circ=2\sin 0^\circ\cos 0^\circ=2\times 0\times 1=0$

$\cos 2\theta=\cos 0^\circ=1-2\sin^2 0^\circ=1-2\times 0=1$

(11)$\theta=30^\circ$

$sin\,\theta = sin\,30° = \frac{1}{2}$，$cos\,\theta = cos\,30° = \frac{\sqrt{3}}{2}$

$sin\,2\theta = sin\,60° = 2\,sin\,30°\,cos\,30° = 2 \times \frac{1}{2} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$

$cos\,2\theta = 1 - 2\,sin^2\,\theta = 1 - 2 \times (\frac{1}{2})^2 = 1 - 2 \times \frac{1}{4} = 1 - \frac{1}{2} = \frac{1}{2}$

(12)$\theta = 45°$

$sin\,45° = \frac{1}{\sqrt{2}}$，$cos\,45° = \frac{1}{\sqrt{2}}$

$sin\,2\theta = sin\,90° = 2\,sin\,45°\,cos\,45° = 2 \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = 2 \times \frac{1}{2} = 1$

$cos\,2\theta = cos\,90° = 1 - 2\,sin^2\,45° = 1 - 2 \times (\frac{1}{\sqrt{2}})^2 = 1 - 2 \times \frac{1}{2} = 1 - 1 = 0$

(13)$\theta = 60°$

$sin\,60° = \frac{\sqrt{3}}{2}$，$cos\,60° = \frac{1}{2}$

$sin\,2\theta = sin\,120° = 2\,sin\,60°\,cos\,60° = 2 \times \frac{\sqrt{3}}{2} \times \frac{1}{2} = \frac{\sqrt{3}}{2}$

$cos\,2\theta = cos\,120° = 1 - 2\,sin^2\,60° = 1 - 2 \times (\frac{\sqrt{3}}{2})^2 = 1 - 2 \times \frac{3}{4} = 1 - \frac{3}{2} = -\frac{1}{2}$

(120 度在第二象限)

(14)$\theta = 90°$

$sin\,90° = 1$，$cos\,90° = 0$

$sin\,2\theta = sin\,180° = 2\,sin\,90°\,cos\,90° = 2 \times 1 \times 0 = 0$

$cos\,2\theta = cos\,180° = 1 - 2\,sin^2\,90° = 1 - 2 \times 1^2 = 1 - 2 = -1$