

## (19)更多的三角函數公式

### 半角公式

$$1. \sin \frac{\theta}{2} = \sqrt{\frac{1-\cos \theta}{2}}$$

證明：

$$\because \cos 2\theta = 1 - 2\sin^2 \theta$$

$$\therefore \cos \theta = 1 - 2\sin^2 \frac{\theta}{2}$$

$$2\sin^2 \frac{\theta}{2} = 1 - \cos \theta$$

$$\sin^2 \frac{\theta}{2} = \frac{1 - \cos \theta}{2}$$

$$\sin \frac{\theta}{2} = \pm \sqrt{\frac{1 - \cos \theta}{2}}$$

(等號右端取正或取負，由 $\frac{\theta}{2}$ 所在的象限決定)

$$2. \cos \frac{\theta}{2} = \pm \sqrt{\frac{1+\cos \theta}{2}}$$

(等號右端取正或取負，由 $\frac{\theta}{2}$ 所在的象限決定)

證明的方法同上

$$3. \tan \frac{\theta}{2} = \pm \sqrt{\frac{1-\cos \theta}{1+\cos \theta}}$$

利用以上兩個公式即可證明

4.  $\theta = 90^\circ$ , 求  $\sin \frac{\theta}{2}$  和  $\cos \frac{\theta}{2}$ 。

$$\cos \theta = \cos 90^\circ = 0$$

$$\sin \frac{\theta}{2} = \sin 45^\circ = \sqrt{\frac{1 - \cos 90^\circ}{2}} = \sqrt{\frac{1 - 0}{2}} = \frac{1}{\sqrt{2}}$$

$$\cos \frac{\theta}{2} = \cos 45^\circ = \sqrt{\frac{1 + \cos 90^\circ}{2}} = \sqrt{\frac{1 + 0}{2}} = \frac{1}{\sqrt{2}}$$

5.  $\theta = 60^\circ$ , 求  $\sin \frac{\theta}{2}$  和  $\cos \frac{\theta}{2}$ 。

$$\cos \theta = \cos 60^\circ = \frac{1}{2}$$

$$\sin \frac{\theta}{2} = \sin 30^\circ = \sqrt{\frac{1 - \cos 60^\circ}{2}} = \sqrt{\frac{1 - \frac{1}{2}}{2}} = \frac{1}{2}$$

$$\cos \frac{\theta}{2} = \cos 30^\circ = \sqrt{\frac{1 + \cos 60^\circ}{2}} = \sqrt{\frac{1 + \frac{1}{2}}{2}} = \frac{\sqrt{3}}{2}$$

### 和積互化公式

6.  $2\sin \alpha \cos \beta = \sin(\alpha + \beta) + \sin(\alpha - \beta)$

證明：

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta \dots(1)$$

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta \dots(2)$$

(1)+(2)得

$$2\sin \alpha \cos \beta = \sin(\alpha + \beta) + \sin(\alpha - \beta)$$

7.  $2\cos \alpha \sin \beta = \sin(\alpha + \beta) - \sin(\alpha - \beta)$

證明方法同 6.

$$8. 2\cos \alpha \cos \beta = \cos(\alpha + \beta) + \cos(\alpha - \beta)$$

證明：

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta \dots(1)$$

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta \dots(2)$$

(1)+(2)得

$$2\cos \alpha \cos \beta = \cos(\alpha + \beta) + \cos(\alpha - \beta)$$

$$9. 2\sin \alpha \sin \beta = \cos(\alpha - \beta) - \cos(\alpha + \beta)$$

證明方法同 8.

$$10. \text{已知 } \alpha = 30^\circ, \beta = 60^\circ, \text{ 驗證 } 2\sin \alpha \cos \beta = \sin(\alpha + \beta) + \sin(\alpha - \beta)$$

過程：

$$\sin \alpha = \sin 30^\circ = \frac{1}{2}$$

$$\cos \beta = \cos 60^\circ = \frac{1}{2}$$

$$2\sin \alpha \cos \beta = \frac{1}{2}$$

$$\sin(\alpha + \beta) = \sin(30^\circ + 60^\circ) = \sin 90^\circ = 1$$

$$\sin(\alpha - \beta) = \sin(30^\circ - 60^\circ) = \sin(-30^\circ) = -\frac{1}{2}$$

$$\sin(\alpha + \beta) + \sin(\alpha - \beta) = 1 + \left(-\frac{1}{2}\right) = \frac{1}{2}$$

因此  $\alpha = 30^\circ, \beta = 60^\circ$  時,  $2\sin \alpha \cos \beta = \sin(\alpha + \beta) + \sin(\alpha - \beta)$

$$11. \text{已知 } \alpha = 60^\circ, \beta = 30^\circ, \text{ 驗證 } 2\cos \alpha \cos \beta = \cos(\alpha + \beta) + \cos(\alpha - \beta)$$

過程：

$$\cos \alpha = \cos 60^\circ = \frac{1}{2}$$

$$\cos \beta = \cos 30^\circ = \frac{\sqrt{3}}{2}$$

$$2\cos \alpha \cos \beta = \frac{\sqrt{3}}{2}$$

$$\cos(\alpha + \beta) = \cos(60^\circ + 30^\circ) = \cos 90^\circ = 0$$

$$\cos(\alpha - \beta) = \cos(60^\circ - 30^\circ) = \cos 30^\circ = \frac{\sqrt{3}}{2}$$

$$\cos(\alpha + \beta) + \cos(\alpha - \beta) = 0 + \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$$

因此  $\alpha = 60^\circ$ ， $\beta = 30^\circ$  時， $2\cos \alpha \cos \beta = \cos(\alpha + \beta) + \cos(\alpha - \beta)$

### 互餘函數的疊合

12. 設  $a, b$  為實數，且  $a^2 + b^2 \neq 0$ ，則：

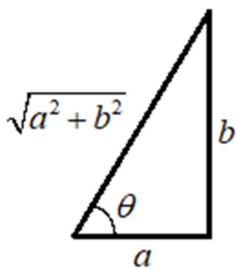
$$y = a \sin x + b \cos x = \sqrt{a^2 + b^2} \sin(x + \theta)$$

$$\theta \text{ 滿足：} \sin \theta = \frac{b}{\sqrt{a^2 + b^2}} \text{、} \cos \theta = \frac{a}{\sqrt{a^2 + b^2}}$$

證明：

$$a \sin x + b \cos x = \sqrt{a^2 + b^2} \frac{a}{\sqrt{a^2 + b^2}} \sin x + \sqrt{a^2 + b^2} \frac{b}{\sqrt{a^2 + b^2}} \cos x$$

我們可以想像有一個直角三角形



在三角形中：

$$\sin \theta = \frac{b}{\sqrt{a^2 + b^2}} \text{、} \cos \theta = \frac{a}{\sqrt{a^2 + b^2}}$$

所以

$$\begin{aligned} & a \sin x + b \cos x \\ &= \sqrt{a^2 + b^2} \frac{a}{\sqrt{a^2 + b^2}} \sin x + \sqrt{a^2 + b^2} \frac{b}{\sqrt{a^2 + b^2}} \cos x \\ &= \sqrt{a^2 + b^2} \cos \theta \sin x + \sqrt{a^2 + b^2} \sin \theta \cos x \\ &= \sqrt{a^2 + b^2} (\cos \theta \sin x + \sin \theta \cos x) \\ &= \sqrt{a^2 + b^2} (\sin(x + \theta)) \end{aligned}$$

可寫成函數形式： $y = a \sin x + b \cos x = \sqrt{a^2 + b^2} \sin(x + \theta)$

13. 將  $y = \frac{\sqrt{3}}{2} \sin x + \frac{1}{2} \cos x$  寫成  $y = \sin(x + \theta)$  的形式，並以  $x = 60^\circ$  驗證。

過程：

$$a = \frac{\sqrt{3}}{2}$$

$$b = \frac{1}{2}$$

$$\sqrt{a^2 + b^2} = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4} + \frac{1}{4}} = 1$$

$$\sin \theta = \frac{b}{\sqrt{a^2 + b^2}} = \frac{1}{2}$$

$$\cos \theta = \frac{a}{\sqrt{a^2 + b^2}} = \frac{\sqrt{3}}{2}$$

滿足此條件的  $\theta$  為  $30^\circ$

因此  $y = \frac{\sqrt{3}}{2} \sin x + \frac{1}{2} \cos x$  可寫成  $y = \sin(x + 30^\circ)$

$x = 60^\circ$  時

$$\frac{\sqrt{3}}{2} \sin 60^\circ + \frac{1}{2} \cos 60^\circ = \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} + \frac{1}{2} \times \frac{1}{2} = 1$$

$$\sin(x + 30^\circ) = \sin(60^\circ + 30^\circ) = \sin 90^\circ = 1$$

可知  $y = \frac{\sqrt{3}}{2} \sin x + \frac{1}{2} \cos x = \sin(x + 30^\circ)$  在  $x = 60^\circ$  時正確。

14. 將  $y = \sin x + \cos x$  寫成  $y = \sin(x + \theta)$  的形式，並以  $x = 45^\circ$  驗證。

過程：

$$a = 1$$

$$b = 1$$

$$\sqrt{a^2 + b^2} = \sqrt{(1)^2 + (1)^2} = \sqrt{2}$$

$$\sin \theta = \frac{b}{\sqrt{a^2 + b^2}} = \frac{1}{\sqrt{2}}$$

$$\cos \theta = \frac{a}{\sqrt{a^2 + b^2}} = \frac{1}{\sqrt{2}}$$

滿足此條件的  $\theta$  為  $45^\circ$

因此  $y = \sin x + \cos x$  可寫成  $y = \sqrt{2} \sin(x + 45^\circ)$

$x = 45^\circ$  時

$$\sin 45^\circ + \cos 45^\circ = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

$$\sin(x + 45^\circ) = \sqrt{2} \sin(45^\circ + 45^\circ) = \sqrt{2} \sin 90^\circ = \sqrt{2}$$

可知  $y = \sin x + \cos x = \sqrt{2} \sin(x + 45^\circ)$  在  $x = 45^\circ$  時正確。

和差化積公式

15.  $\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$ , 求證:

我們已知

$$\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2 \sin \alpha \cos \beta$$

令  $A = \alpha + \beta$

$$B = \alpha - \beta$$

$$\alpha = \frac{A + B}{2}$$

$$\beta = \frac{A - B}{2}$$

$$\therefore \sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$$

故得證

16. 用同樣的方法, 可以得到

$$\sin A - \sin B = 2 \cos \frac{A + B}{2} \sin \frac{A - B}{2}$$

$$\cos A + \cos B = 2 \cos \frac{A + B}{2} \cos \frac{A - B}{2}$$

$$\cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}$$

17. 驗證  $\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$

令  $A=0^\circ, B=60^\circ$

$$\sin A + \sin B = \sin 0^\circ + \sin 60^\circ = \frac{\sqrt{3}}{2}$$

$$\sin \frac{A+B}{2} = \sin 30^\circ = \frac{1}{2}$$

$$\cos \frac{A-B}{2} = \cos(-30^\circ) = \frac{\sqrt{3}}{2}$$

$$\therefore 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2} = 2 \times \frac{1}{2} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$$

$$\therefore \sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$$

18. 驗證  $\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$

令  $A=60^\circ, B=0^\circ$

$$\cos A = \cos 60^\circ = \frac{1}{2}$$

$$\cos B = \cos 0^\circ = 1$$

$$\therefore \cos A + \cos B = \frac{1}{2} + 1 = \frac{3}{2}$$

$$\cos \frac{A+B}{2} = \cos 30^\circ = \frac{\sqrt{3}}{2}$$

$$\cos \frac{A-B}{2} = \cos 30^\circ = \frac{\sqrt{3}}{2}$$

$$2 \cos \frac{A+B}{2} \cos \frac{A-B}{2} = 2 \times \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} = \frac{3}{2}$$

$$\therefore \cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$$