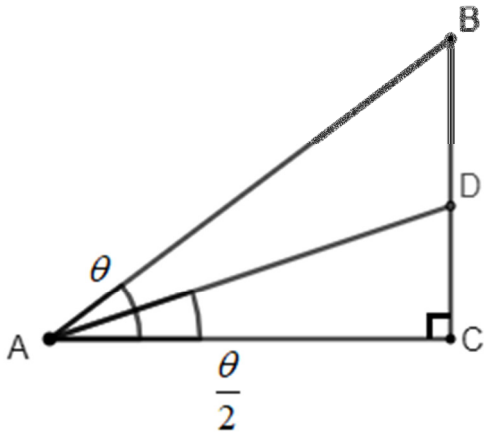


## (17) 半角的三角函數

### 半角的三角函數



$\triangle ABC$  中， $\angle C$  為直角， $D$  點在  $\overline{BC}$  上，且  $\overline{AD}$  為  $\angle A$  的角平分線。

$$\angle A = \theta, \quad \angle CAD = \frac{\theta}{2}$$

$$\text{證明 } \sin \frac{\theta}{2} = \frac{\sin \theta}{\sqrt{2 \times (1 + \cos \theta)}}$$

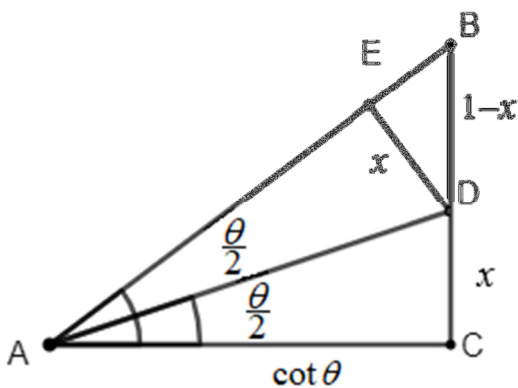
過程：

假設  $\overline{BC} = 1$ ，

$$\frac{\overline{AC}}{\overline{BC}} = \cot \theta, \quad \overline{AC} = \cot \theta$$

在  $\overline{AB}$  上取一點  $E$ ，且  $\overline{DE} \perp \overline{AB}$ 。

設  $\overline{DC} = x$ ，則  $\overline{BD} = 1 - x$



$$\because \angle DAC = \angle DAE = \frac{\theta}{2}, \overline{AD} = \overline{AD}, \angle ADC = \angle ADE = 90^\circ - \frac{\theta}{2}$$

$$\therefore \triangle AED \cong \triangle ACD \text{ (ASA)}$$

$$\therefore \overline{ED} = \overline{DC} = x$$

$$\angle B = 90^\circ - \angle A = 90^\circ - \theta$$

$$\text{由 } \triangle BED \text{ 可以看出 } \sin B = \frac{x}{1-x}$$

$$\sin B = \sin(90^\circ - \theta) = \cos \theta$$

$$\therefore \cos \theta = \frac{x}{1-x}$$

$$x = \frac{\cos \theta}{1 + \cos \theta} \dots (1)$$

$\triangle ADC$  中

$$\overline{AD}^2 = x^2 + \cot^2 \theta$$

$$\overline{AD}^2 = \left(\frac{\cos \theta}{1 + \cos \theta}\right)^2 + \cot^2 \theta$$

$$\overline{AD}^2 = \frac{\cos^2 \theta}{(1 + \cos \theta)^2} + \frac{\cos^2 \theta}{\sin^2 \theta}$$

$$\overline{AD}^2 = \frac{\cos^2 \theta \sin^2 \theta + \cos^2 \theta (1 + \cos \theta)^2}{(1 + \cos \theta)^2 \sin^2 \theta}$$

$$\overline{AD}^2 = \frac{\cos^2 \theta (\sin^2 \theta + (1 + \cos \theta)^2)}{(1 + \cos \theta)^2 \sin^2 \theta}$$

$$\overline{AD}^2 = \frac{\cos^2 \theta (\sin^2 \theta + 1 + 2 \cos \theta + \cos^2 \theta)}{(1 + \cos \theta)^2 \sin^2 \theta}$$

$$\overline{AD}^2 = \frac{\cos^2 \theta (2 + 2 \cos \theta)}{(1 + \cos \theta)^2 \sin^2 \theta}$$

$$\overline{AD}^2 = \frac{2 \cos^2 \theta}{(1 + \cos \theta) \sin^2 \theta}$$

$$\therefore \overline{AD} = \frac{\sqrt{2} \cos \theta}{\sin \theta \sqrt{1 + \cos \theta}} \dots (2)$$

$$\sin \frac{\theta}{2} = \frac{x}{\overline{AD}}$$

$$\sin \frac{\theta}{2} = \frac{\frac{\cos \theta}{1 + \cos \theta}}{\frac{\sqrt{2} \cos \theta}{\sin \theta \sqrt{1 + \cos \theta}}}$$

$$\sin \frac{\theta}{2} = \frac{\frac{\cos \theta}{1 + \cos \theta}}{\frac{\sqrt{2} \cos \theta}{\sin \theta \sqrt{1 + \cos \theta}}}$$

$$\sin \frac{\theta}{2} = \frac{\cos \theta}{1 + \cos \theta} \times \frac{\sin \theta \sqrt{1 + \cos \theta}}{\sqrt{2} \cos \theta}$$

$$\sin \frac{\theta}{2} = \frac{\sin \theta}{\sqrt{2} \sqrt{1 + \cos \theta}}$$

例 1

$$\theta = 60^\circ, \text{ 求 } \sin \frac{\theta}{2}$$

詳解：

$$\sin \frac{\theta}{2} = \sin \frac{60^\circ}{2} = \frac{\sin 60^\circ}{\sqrt{2} \sqrt{1 + \cos 60^\circ}} = \frac{\frac{\sqrt{3}}{2}}{\sqrt{2} \sqrt{1 + \frac{1}{2}}} = \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{3}} = \frac{1}{2}$$

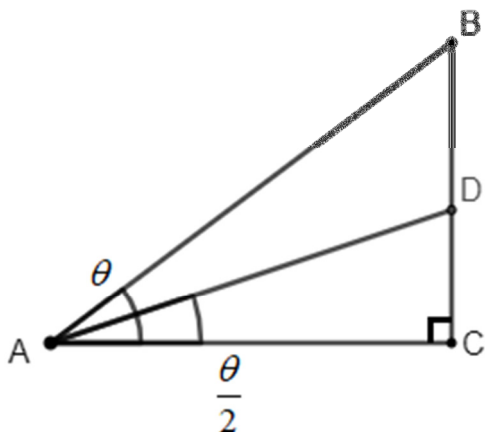
例 2

$$\theta = 90^\circ, \text{ 求 } \sin \frac{\theta}{2}$$

詳解：

$$\sin \frac{\theta}{2} = \sin \frac{90^\circ}{2} = \frac{\sin 90^\circ}{\sqrt{2} \sqrt{1 + \cos 90^\circ}} = \frac{1}{\sqrt{2} \sqrt{1 + 0}} = \frac{1}{\sqrt{2}}$$

我們找到了求 $\sin \frac{\theta}{2}$ 的方式，就可以很容易地找到求 $\cos \frac{\theta}{2}$ 的方式。



求 $\sin \frac{\theta}{2}$ 時我們已經知道 $\overline{AD} = \frac{\sqrt{2} \cos \theta}{\sin \theta \sqrt{1 + \cos \theta}}$

$$\cos \frac{\theta}{2} = \frac{\overline{AC}}{\overline{AD}}$$

$$\cos \frac{\theta}{2} = \frac{\cot \theta}{\frac{\sqrt{2} \cos \theta}{\sin \theta \sqrt{1 + \cos \theta}}}$$

$$\cos \frac{\theta}{2} = \frac{\cot \theta}{\frac{\sqrt{2} \cot \theta}{\sqrt{1 + \cos \theta}}}$$

$$\cos \frac{\theta}{2} = \frac{\sqrt{1 + \cos \theta}}{\sqrt{2}}$$

我們已經知道 $\sin \frac{\theta}{2} = \frac{\sin \theta}{\sqrt{2} \sqrt{1 + \cos \theta}}$ 、 $\cos \frac{\theta}{2} = \frac{\sqrt{1 + \cos \theta}}{\sqrt{2}}$

$$\text{所以 } \sin \frac{\theta}{2} \times \cos \frac{\theta}{2} = \frac{\sin \theta}{2}$$

$$\sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}$$

我們也可以寫成  $\sin 2\theta = 2 \sin \theta \cos \theta$

例 3

$\theta = 30^\circ$ ，求  $\sin 2\theta$

詳解：

$$\sin 2\theta = \sin(2 \times 30^\circ) = 2 \sin 30^\circ \cos 30^\circ = 2 \times \frac{1}{2} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$$

例 4

$$\theta = 45^\circ, \text{ 求 } \sin 2\theta$$

詳解：

$$\sin 2\theta = \sin(2 \times 45^\circ) = 2 \sin 45^\circ \cos 45^\circ = 2 \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = 1$$

例 5

$$\theta = 15^\circ, \text{ 求 } \sin 2\theta$$

詳解：

之前我們學過

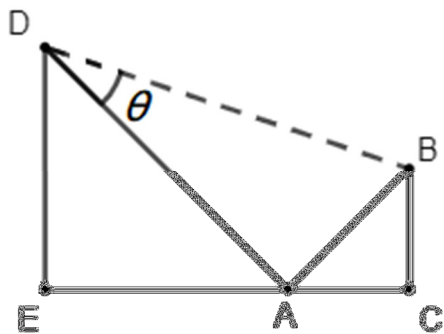
$$\sin 15^\circ = \frac{\sqrt{6} - \sqrt{2}}{4}$$

$$\cos 15^\circ = \frac{\sqrt{6} + \sqrt{2}}{4}$$

$$\sin 2\theta = \sin(2 \times 15^\circ) = 2 \sin 15^\circ \cos 15^\circ = 2 \times \frac{\sqrt{6} - \sqrt{2}}{4} \times \frac{\sqrt{6} + \sqrt{2}}{4} = \frac{6 - 2}{8} = \frac{1}{2}$$

### 三角形中角的三角函數

1.



上圖中， $\angle E = \angle C = 90^\circ$ ， $\overline{EA} = \overline{ED} = 2$ ， $\overline{CA} = \overline{CB} = 1$ ，求  $\sin \theta$  之值。

詳解：

$$\overline{CA} = \overline{CB}, \angle C = 90^\circ$$

$$\therefore \angle BAC = 45^\circ$$

$$\text{同理 } \angle DAE = 45^\circ$$

$$\therefore \angle DAB = 180^\circ - 45^\circ - 45^\circ = 90^\circ$$

$$\therefore \sin \theta = \frac{\overline{AB}}{\overline{BD}}$$

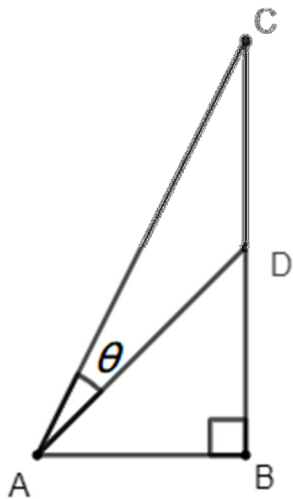
$$\overline{AB} = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\overline{AD} = \sqrt{2^2 + 2^2} = \sqrt{8}$$

$$\overline{BD} = \sqrt{\sqrt{2}^2 + \sqrt{8}^2} = \sqrt{10}$$

$$\sin \theta = \frac{\overline{AB}}{\overline{BD}} = \frac{\sqrt{2}}{\sqrt{10}} = \frac{1}{\sqrt{5}}$$

2.

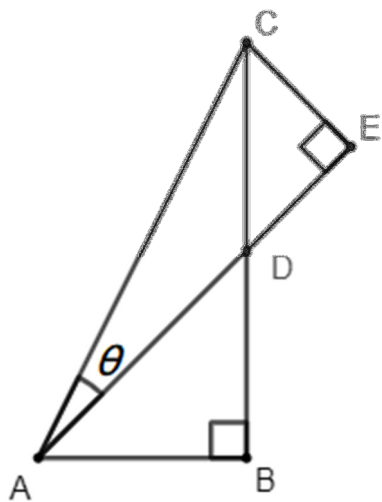


上圖中， $\overline{AB}=1$ ， $\overline{BC}=2$ ， $\overline{CD}=\overline{DB}=1$ ，求  $\sin \theta$  之值。

詳解：

$$\overline{AC} = \sqrt{\overline{AB}^2 + \overline{BC}^2} = \sqrt{1^2 + 2^2} = \sqrt{5}$$

延長  $\overline{AD}$  至 E 點，且  $\overline{AE} \perp \overline{EC}$



$\overline{AB} = \overline{DB} = 1$ ， $\triangle ABD$  是等腰直角三角形

$$\angle BAD = \angle BDA = 45^\circ$$

$$\angle CDE = \angle BDA = 45^\circ \text{ (對頂角相等)}$$

$$\angle DCE = 90^\circ - 45^\circ = 45^\circ$$

$$\angle DCE = \angle CDE$$

$$\therefore \overline{CE} = \overline{DE}$$

$$\overline{CD}^2 = 1 = \overline{CE}^2 + \overline{DE}^2 = 2\overline{CE}^2$$

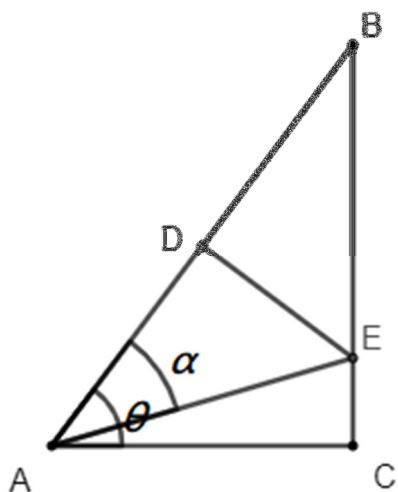
$$\overline{CE}^2 = \frac{1}{2}$$

$$\overline{CE} = \frac{1}{\sqrt{2}}$$

$\triangle ACE$  中

$$\sin \theta = \frac{\overline{CE}}{\overline{AC}} = \frac{\frac{1}{\sqrt{2}}}{\sqrt{5}} = \frac{1}{\sqrt{10}}$$

3.



上圖中， $\angle BAC = \theta$ ， $\angle BAE = \alpha$ ， $\overline{DE}$  是  $\overline{AB}$  的中垂線，將  $\sin \alpha$  用  $\theta$  表示。

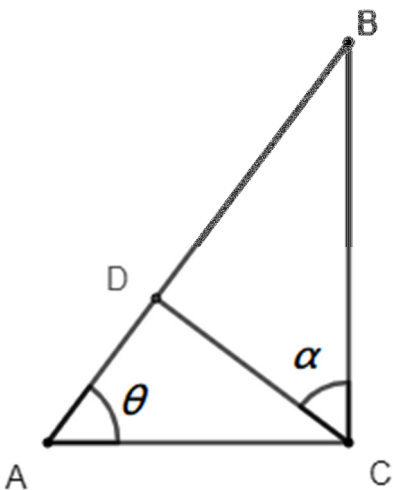
詳解：

依題意  $\triangle ADE \cong \triangle BDE$  (SAS)

$$\therefore \alpha = \angle EBA = \angle CBA = 90^\circ - \theta$$

$$\sin \alpha = \sin(90^\circ - \theta) = \cos \theta$$

4.



上圖中， $\angle BCA = 90^\circ$ ， $\overline{AB} \perp \overline{DC}$ ， $\angle BAC = \theta$ ， $\angle BCD = \alpha$ ，將  $\sin \alpha$  用  $\theta$  表示。

詳解：

$$\triangle BCD \text{ 中，} \angle B + \angle \alpha = 90^\circ$$

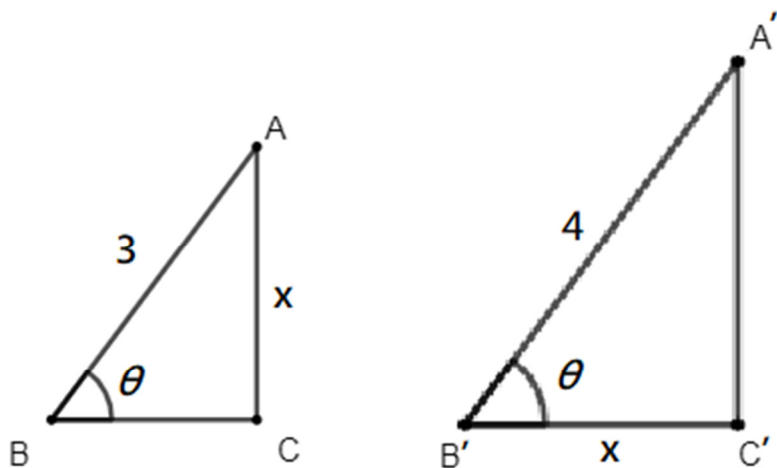


$\triangle ABC$  中， $\angle B + \angle \theta = 90^\circ$

可知  $\alpha = \theta$

$$\sin \alpha = \sin \theta$$

5.



上圖兩中角形中， $\angle C = \angle C' = 90^\circ$ ，求  $x$  之值。

詳解：

$$\triangle ABC \text{ 中，} \sin \theta = \frac{x}{3} \dots (1)$$

$\triangle A'B'C'$  中

$$\overline{A'C'} = \sqrt{4^2 - x^2} = \sqrt{16 - x^2}$$

$$\sin \theta = \frac{\sqrt{16 - x^2}}{4} \dots (2)$$

由(1)(2)可知

$$\frac{x}{3} = \frac{\sqrt{16 - x^2}}{4}$$

$$4x = 3\sqrt{16 - x^2}$$

$$16x^2 = 9(16 - x^2)$$

$$16x^2 = 144 - 9x^2$$

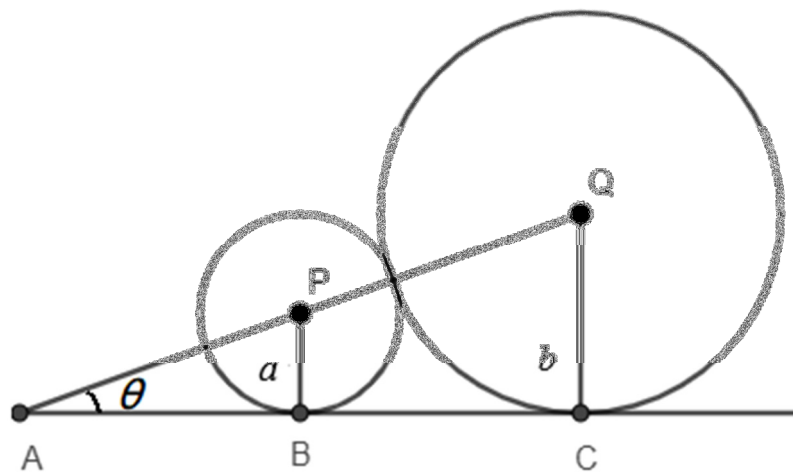
$$25x^2 = 144$$

$$x^2 = \frac{144}{25}$$

$$x = \frac{12}{5}$$

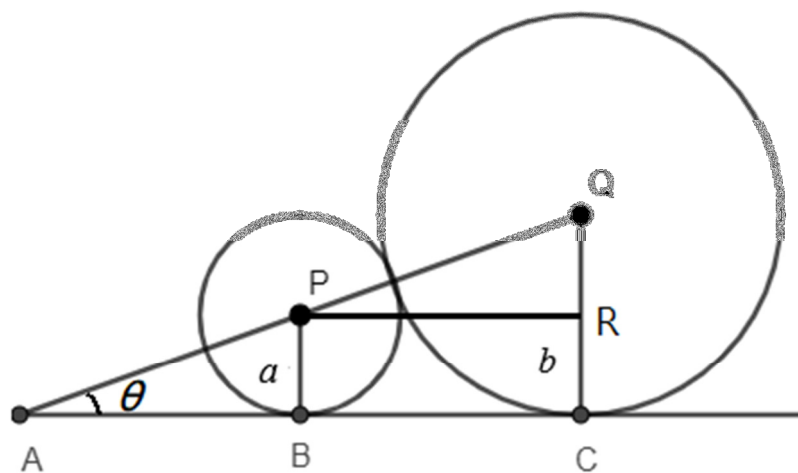
(長度為正數，取正值)

6.



如上圖， $P$ 、 $Q$  兩圓相切，半徑分別為  $a$ 、 $b$ ， $a < b$ ， $\overline{AC}$  為  $P$ 、 $Q$  的切線，切點分別為  $B$ 、 $C$ 。求  $\tan \theta$ 。

詳解：



從  $P$  做  $\overline{BC}$  之平行線，與  $\overline{QC}$  交於  $R$  點。

$\angle QPR = \theta$  (同位角相等)

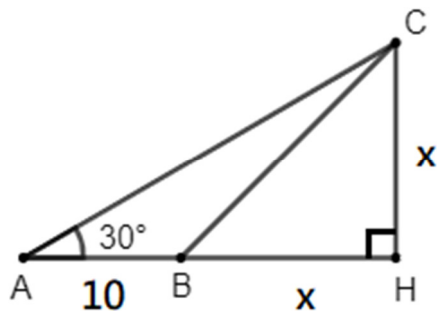
$$\overline{PQ} = a + b$$

$$\overline{QR} = b - a$$

$$\overline{PR} = \sqrt{(a + b)^2 - (b - a)^2} = \sqrt{4ab} = 2\sqrt{ab}$$

$$\tan \theta = \frac{\overline{QR}}{\overline{PR}} = \frac{b - a}{2\sqrt{ab}}$$

7.



如上圖，求  $x$  之值。

詳解：

$AHC$  為  $30^\circ$ - $60^\circ$ - $90^\circ$  的直角三角形

$$\sin A = \sin 30^\circ = \frac{1}{2} = \frac{x}{\overline{AC}}$$

$$\text{故 } \overline{AC} = 2x$$

$$\cos 30^\circ = \frac{\sqrt{3}}{2} = \frac{\overline{AH}}{\overline{AC}} = \frac{10 + x}{2x}$$

$$\frac{\sqrt{3}}{2} = \frac{10 + x}{2x}$$

$$\sqrt{3} = \frac{10 + x}{x}$$

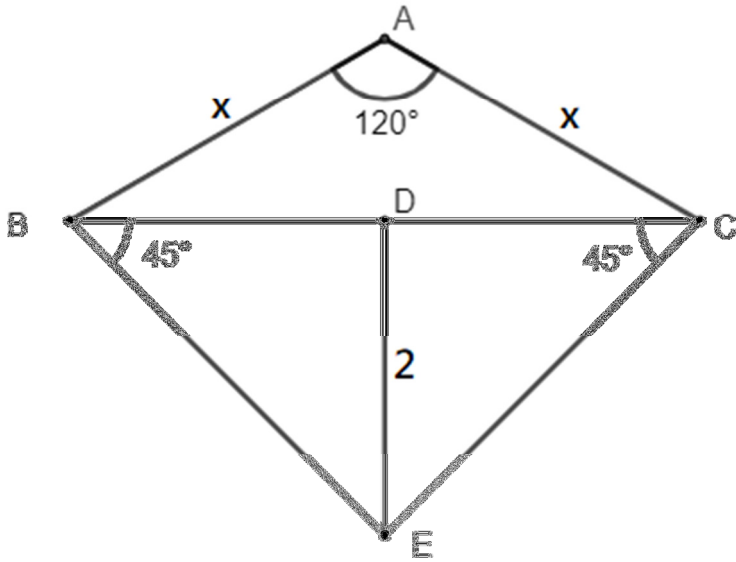
$$\sqrt{3}x = 10 + x$$

$$\sqrt{3}x - x = 10$$

$$(\sqrt{3} - 1)x = 10$$

$$x = \frac{10}{\sqrt{3} - 1}$$

8.



如上圖，求  $x$  之值。

詳解：

$$\overline{BC} = \sqrt{x^2 + x^2 - 2x^2 \cos 120^\circ} = \sqrt{2x^2 - 2x^2 \left(-\frac{1}{2}\right)} = \sqrt{3x^2} = \sqrt{3}x$$

$$\overline{BD} = \frac{1}{2}\overline{BC} = \frac{\sqrt{3}}{2}x$$

因為  $\angle DBE = 45^\circ$ ， $\overline{BD} = \overline{DE} = \frac{\sqrt{3}}{2}x = 2$

$$x = \frac{4}{\sqrt{3}}$$