(15) 三角總複習 1

1. 在ΔABC中，求證：$\sin A = \sin(B + C)$

【證明】$\because \angle A = 180° - (\angle B + \angle C)$

$\therefore \sin A = \sin(180° - (B + C)) = \sin(B + C)$

2. 在ΔABC中，求證：$\sin\frac{B+C-A}{2} = \cos A$

【證明】

$\because B + C = 180° - A$

$\therefore \sin\dfrac{B + C - A}{2}$

$= \sin\dfrac{180° - 2A}{2}$

$= \sin(90° - A)$

$= \cos A$

3. 已知$\tan\theta = a$，求$\cos\theta$

【解答】$\because 1 + \tan^2\theta = \sec^2\theta$

$1 + \tan^2\theta = \frac{1}{\cos^2\theta}$

$\therefore \cos^2\theta = \frac{1}{1+\tan^2\theta} = \frac{1}{1+a^2}$

$\cos\theta = \frac{1}{\sqrt{1+a^2}}$

4. 求證：$\frac{1}{1+\sin\theta} + \frac{1}{1-\sin\theta} = 2\sec^2\theta$

【證明】$\because \frac{1}{1+\sin\theta}+\frac{1}{1-\sin\theta}$

$$=\frac{1-\sin\theta+1+\sin\theta}{(1+\sin\theta)(1-\sin\theta)}=\frac{2}{1-sin^2\theta}=\frac{2}{cos^2\theta}=2\,sec^2\theta$$

5. 求證：$(1-\tan^4\theta)\,cos^2\theta+\tan^2\theta=1$

【證明】 $(1-\tan^4\theta)\,cos^2\theta+\tan^2\theta$

$$=(1-\tan^2\theta)(1+\tan^2\theta)\,cos^2\theta+\tan^2\theta$$
$$=(1-\tan^2\theta)\,sec^2\theta\,cos^2\theta+\tan^2\theta$$
$$=(1-\tan^2\theta)\frac{1}{cos^2\theta}cos^2\theta+\tan^2\theta$$
$$=(1-\tan^2\theta)+\tan^2\theta$$
$$=1$$

6. 求證：$\frac{cos\,\theta}{1+\sin\theta}+\frac{cos\,\theta}{1-\sin\theta}=2\,sec\,\theta$

【證明】 $\frac{cos\,\theta}{1+\sin\theta}+\frac{cos\,\theta}{1-\sin\theta}$

$$=\frac{cos\,\theta\,(1-\sin\theta)+cos\,\theta\,(1+\sin\theta)}{(1+\sin\theta)(1-\sin\theta)}$$
$$=\frac{2\,cos\,\theta}{1-sin^2\theta}$$
$$=\frac{2\,cos\,\theta}{cos^2\theta}$$
$$=\frac{2}{cos\,\theta}$$
$$=2\,sec\,\theta$$

7. 求證：

$$\frac{cos\,\theta}{1+\sin\theta}+\frac{1+\sin\theta}{cos\,\theta}=2\,sec\,\theta$$

【證明】

$$\frac{cos\,\theta}{1+\sin\theta}+\frac{1+\sin\theta}{cos\,\theta}$$

$$= \frac{cos^2\theta + (1+\sin\theta)^2}{(1+\sin\theta)\,cos\,\theta}$$
$$= \frac{cos^2\theta + 1 + 2\sin\theta + \sin^2\theta}{(1+\sin\theta)\,cos\,\theta}$$
$$= \frac{1+1+2\sin\theta}{(1+\sin\theta)\,cos\,\theta}$$
$$= \frac{2(1+\sin\theta)}{(1+\sin\theta)\,cos\,\theta}$$
$$= \frac{2}{cos\,\theta} = 2\,sec\,\theta$$

8. 求證：

$$\frac{sin\,\theta - cos\,\theta}{\sin\theta + cos\,\theta} + \frac{tan\,\theta + 1}{tan\,\theta - 1} = \frac{2}{\sin^2\theta - cos^2\theta}$$

【證明】

$$\because \frac{tan\,\theta + 1}{tan\,\theta - 1} = \frac{\frac{sin\,\theta}{cos\,\theta} + 1}{\frac{sin\,\theta}{cos\,\theta} - 1} = \frac{\frac{sin\,\theta + cos\,\theta}{cos\,\theta}}{\frac{sin\,\theta - cos\,\theta}{cos\,\theta}} = \frac{sin\,\theta + cos\,\theta}{sin\,\theta - cos\,\theta}$$
$$\therefore \frac{sin\,\theta - cos\,\theta}{\sin\theta + cos\,\theta} + \frac{tan\,\theta + 1}{tan\,\theta - 1}$$
$$= \frac{sin\,\theta - cos\,\theta}{\sin\theta + cos\,\theta} + \frac{sin\,\theta + cos\,\theta}{sin\,\theta - cos\,\theta}$$
$$= \frac{(sin\,\theta - cos\,\theta)^2 + (sin\,\theta + cos\,\theta)^2}{(\sin\theta + cos\,\theta)(sin\,\theta - cos\,\theta)}$$
$$= \frac{2(sin^2\theta + cos^2\theta)}{sin^2\theta - cos^2\theta}$$
$$= \frac{2}{\sin^2\theta - cos^2\theta}$$

9. 求證：$(\sin\theta - \frac{1}{\sin\theta})^2 - (tan\,\theta - \frac{1}{tan\,\theta})^2 + (cos\,\theta - \frac{1}{cos\,\theta})^2 = 1$

【證明】$(\sin\theta - \frac{1}{\sin\theta})^2 - (tan\,\theta - \frac{1}{tan\,\theta})^2 + (cos\,\theta - \frac{1}{cos\,\theta})^2$

$= (\sin^2\theta + \frac{1}{\sin^2\theta} - 2) - (tan^2\theta + \frac{1}{tan^2\theta} - 2) + (cos^2\theta +$

$\frac{1}{cos^2\theta}-2)$

$=\sin^2\theta+\frac{1}{\sin^2\theta}-2-\frac{\sin^2\theta}{cos^2\theta}-\frac{cos^2\theta}{\sin^2\theta}+2+cos^2\theta+\frac{1}{cos^2\theta}-2$

$=(\sin^2\theta+cos^2\theta)+\frac{1-cos^2\theta}{\sin^2\theta}+\frac{1-\sin^2\theta}{cos^2\theta}-2$

$=1+\frac{sin^2\theta}{\sin^2\theta}+\frac{cos^2\theta}{cos^2\theta}-2$

$=1+1+1-2$

$=1$

10. 求證：

$$\frac{1+sin\,\theta}{1-\sin\theta}=(\frac{cos\,\theta}{1-\sin\theta})^2$$

【證明】

$$\frac{1+sin\,\theta}{1-\sin\theta}$$

$$=\frac{(1+sin\,\theta)(1-\sin\theta)}{(1-\sin\theta)(1-\sin\theta)}$$

$$=\frac{1-\sin^2\theta}{(1-\sin\theta)^2}$$

$$=\frac{cos^2\theta}{(1-\sin\theta)^2}$$

$$=(\frac{cos\,\theta}{1-\sin\theta})^2$$

11. 求證：$(1+sin\,\theta+cos\,\theta)(1+sin\,\theta-cos\,\theta)=2\,sin\,\theta\,(1+sin\,\theta)$

【證明】$(1+sin\,\theta+cos\,\theta)(1+sin\,\theta-cos\,\theta)$

$=(1+sin\,\theta)^2-cos^2\theta$

$$= (1 + \sin\theta)^2 - (1 - \sin^2\theta)$$
$$= (1 + \sin\theta)^2 - (1 + \sin\theta)(1 - \sin\theta)$$
$$= (1 + \sin\theta)(1 + \sin\theta - 1 + \sin\theta)$$
$$= 2\sin\theta\,(1 + \sin\theta)$$

12. 求證：

$$\frac{(1+tan\theta)^2}{(1-tan\theta)^2} = \frac{1+2sin\theta cos\theta}{1-2sin\theta cos\theta}$$

【證明】

$$\frac{(1+tan\theta)^2}{(1-tan\theta)^2}$$
$$= \frac{\left(1+\frac{sin\theta}{cos\theta}\right)^2}{\left(1-\frac{sin\theta}{cos\theta}\right)^2}$$
$$= \frac{\left(\frac{cos\theta+sin\theta}{cos\theta}\right)^2}{\left(\frac{cos\theta-sin\theta}{cos\theta}\right)^2}$$
$$= \left(\frac{cos\theta+sin\theta}{cos\theta-sin\theta}\right)^2$$
$$= \frac{cos^2\theta+sin^2\theta+2sin\theta cos\theta}{cos^2\theta+sin^2\theta-2sin\theta cos\theta}$$
$$= \frac{1+2sin\theta cos\theta}{1-2sin\theta cos\theta}$$

13. 已知$\begin{cases} x = \sin\theta - 2 \ldots\ldots (1) \\ y = \cos\theta + 2 \ldots\ldots (2) \end{cases}$，求證：$(x+2)^2 + (y-2)^2 = 1$

【證明】由(1)得 $x + 2 = \sin\theta$

由(2)得 $y - 2 = \cos\theta$

$\therefore (x+2)^2 + (y-2)^2 = \sin^2\theta + \cos^2\theta = 1$

14. 求證：$1 - sin\,\theta + sin^2\,\theta - cos^2\,\theta = sin\,\theta\,(2\,sin\,\theta - 1)$

【證明】$1 - sin\,\theta + sin^2\,\theta - cos^2\,\theta$

$= sin^2\,\theta + cos^2\,\theta - sin\,\theta + sin^2\,\theta - cos^2\,\theta$

$= 2\,sin^2\,\theta - sin\,\theta$

$= sin\,\theta\,(2\,sin\,\theta - 1)$

15. 求證：$1 + (sin\,\theta - cos\,\theta)^2 = 2(1 - sin\,\theta\,cos\,\theta)$

【證明】$1 + (sin\,\theta - cos\,\theta)^2$

$= 1 + sin^2\,\theta + cos^2\,\theta - 2\,sin\,\theta\,cos\,\theta$

$= 1 + 1 - 2\,sin\,\theta\,cos\,\theta$

$= 2 - 2\,sin\,\theta\,cos\,\theta$

$= 2(1 - sin\,\theta\,cos\,\theta)$

16. 已知$\begin{cases} sin\,\theta + cos\,\theta = -k \ldots\ldots (1) \\ sin\,\theta\,cos\,\theta = k \ldots\ldots\ldots\ldots (2) \end{cases}$，求 k

【解答】由(1)得

$(1)^2 \Rightarrow (sin\,\theta + cos\,\theta)^2 = (-k)^2$

$\Rightarrow sin^2\,\theta + cos^2\,\theta + 2\,sin\,\theta\,cos\,\theta = k^2$

$\Rightarrow 1 + 2\,sin\,\theta\,cos\,\theta = k^2 \ldots\ldots. (3)$

將(2)代入(3)得

$1 + 2k = k^2$

$k^2 - 2k - 1 = 0$

$k = \dfrac{2 \pm \sqrt{4+4}}{2}$

$= \dfrac{2 \pm 2\sqrt{2}}{2}$

$= 1 \pm \sqrt{2}$

$\mathrm{k} = 1 + \sqrt{2} > 1$，不合(2)

$\because -1 < sin\,\theta < 1$

$-1 < cos\,\theta < 1$

$\Rightarrow -1 < sin\,\theta\,cos\,\theta < 1$

$\Rightarrow -1 < k < 1$

$\therefore k = 1 - \sqrt{2}$

17. 求證：$(cos\,A\,(1+tan\,A))^2+(cos\,A\,(1-tan\,A))^2=2$

【證明】$(cos\,A\,(1+tan\,A))^2+(cos\,A\,(1-tan\,A))^2$
$=cos^2\,A\,(1+tan\,A)^2+cos^2\,A\,(1-tan\,A)^2$
$=cos^2\,A\,(1+2\,tan\,A+tan^2\,A)+cos^2\,A\,(1-2\,tan\,A+tan^2\,A)$
$=cos^2\,A\,(1+2\,tan\,A+tan^2\,A+1-2\,tan\,A+tan^2\,A)$
$=cos^2\,A\,(2+2\,tan^2\,A)$
$=2\,cos^2\,A\,(1+tan^2\,A)$
$=2\,cos^2\,A\,sec^2\,A$
$=2$