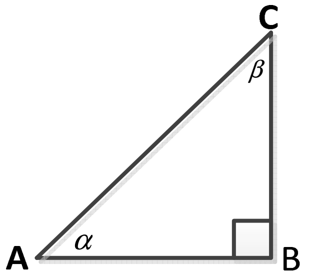
(11)三角函數的轉換

同學們一定對以下的知識很熟悉:

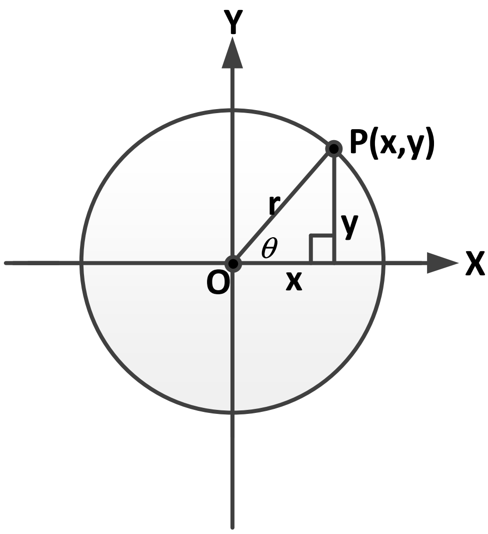


因為$\alpha+\beta=90°$，所以它們有以下的關係:

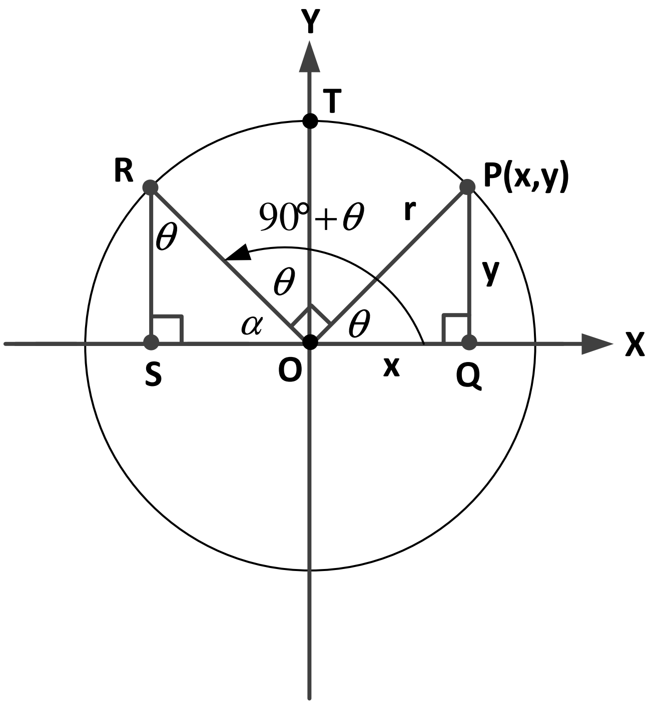
$$\sin\alpha=\frac{\overline{BC}}{\overline{AB}}=\cos\beta$$

$$\cos\alpha=\frac{\overline{AC}}{\overline{AB}}=\sin\beta$$

我們先假設θ是一銳角。



(1)$90°+\theta$



$\angle TOR = \angle POQ = \theta$

$\angle POR = 90°$

$\angle QOR = 90° + \theta$

$\because RS \parallel OT$， $\therefore \angle ORS = \theta$

考慮ΔOPQ 和ΔORS

$\angle ROS + \angle POQ = 90°$

而$\angle OPQ + \angle POQ = 90°$

$\therefore \angle ROS = \angle OPQ$

$\because RS \parallel OT$

$\angle ORS = \theta = \angle POQ$

$\overline{OR} = \overline{OP} = r$

$\therefore \Delta OPQ \cong \Delta ORS$ (ASA)

$\therefore\ \overline{RS} = \overline{OQ} = x$

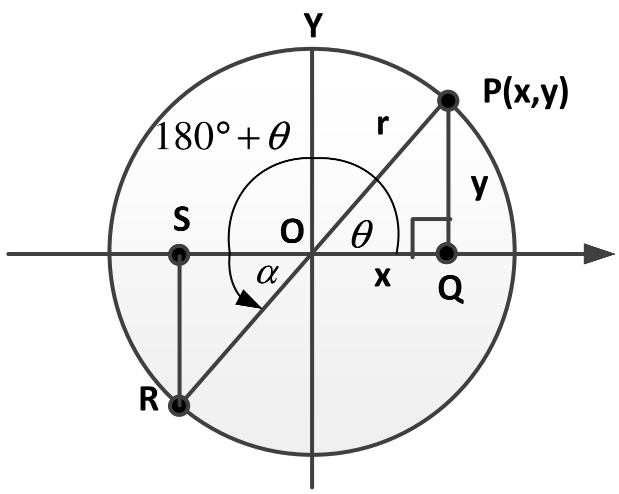
$\overline{OS} = \overline{PQ} = y$

$$\sin(90° + \theta) = \sin\alpha = \frac{\overline{RS}}{r} = \frac{x}{r} = \cos\theta$$

$$\cos(90° + \theta) = \cos\alpha = \frac{\overline{OS}}{r} = \frac{-y}{x} = -\sin\theta \ \ (x\text{在第二象限為負})$$

$$\tan(90° + \theta) = \tan\alpha = \frac{\overline{RS}}{\overline{OS}} = \frac{x}{-y} = -\cot\theta$$

（2）$180° + \theta$



$\angle ROS = \alpha = \angle POQ = \theta$ (對頂角)

$\angle QOR = 180° + \theta$

我們可以證明 $\Delta OPQ \equiv \Delta ORS$

$\therefore\ \overline{OS} = \overline{OQ} = x$

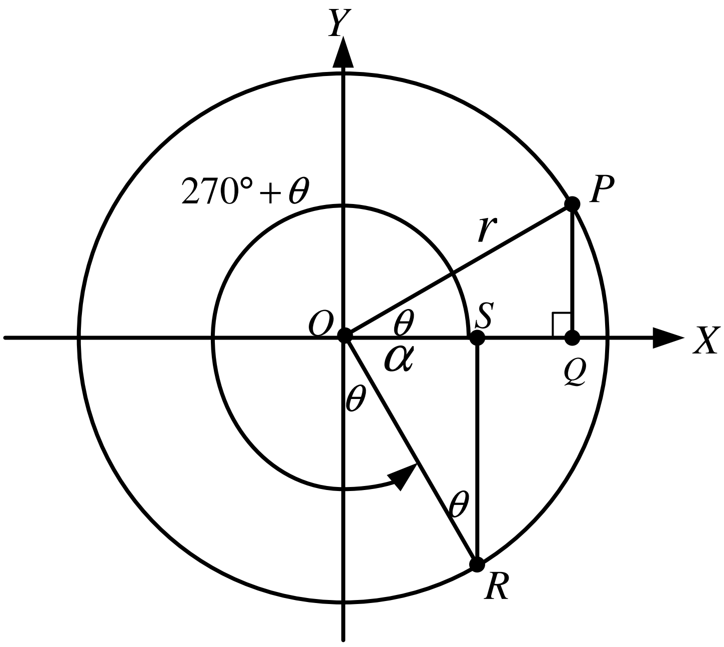
$\overline{RS} = \overline{PQ} = y$

$\sin(180° + \theta) = \sin\alpha = \frac{\overline{RS}}{r} = \frac{-y}{r} = -\sin\theta$ (y在第三象限為負)

$\cos(180° + \theta) = \cos\alpha = \frac{\overline{OS}}{r} = \frac{-x}{r} = -\cos\theta$ (x在第三象限為負)

$\tan(180° + \theta) = \tan\alpha = \frac{\overline{RS}}{\overline{OS}} = \frac{-y}{-x} = \tan\theta$

(3) $270° + \theta$



$\angle ROT = \theta$

$\angle POR = 270° + \theta$

我們可以證明 $\Delta OPQ \equiv \Delta ORS$

$\overline{OS} = \overline{PQ} = y$

$\overline{RS} = \overline{OQ} = x$

$\sin(270° + \theta) = \sin\alpha = \frac{\overline{RS}}{r} = \frac{-x}{r} = -\cos\theta$ (y在第四象限為負)

$$\cos(270° + \theta) = \cos\alpha = \frac{\overline{OS}}{r} = \frac{y}{r} = \sin\theta$$

$$\tan(270° + \theta) = \tan\alpha = \frac{\overline{RS}}{\overline{OS}} = \frac{-x}{y} = -\cot\theta$$

我們將以上的公式整理如下:

$$\sin(90° + \theta) = \cos\theta$$

$$\sin(180° + \theta) = -\sin\theta$$

$$\sin(270° + \theta) = -\cos\theta$$

$$\cos(90° + \theta) = -\sin\theta$$

$$\cos(180° + \theta) = -\cos\theta$$

$$\cos(270° + \theta) = \sin\theta$$

$$\tan(90° + \theta) = -\cot\theta$$

$$\tan(180° + \theta) = \tan\theta$$

$$\tan(270° + \theta) = -\cot\theta$$

因此，我們不必將θ限於銳角，而使得三角函數的角可以任意象限的角，也就是所謂廣義的三角函數。

我們的結論可以用以下的表表示

	第 1 象限	第 2 象限	第 3 象限	第 4 象限
$\sin\theta$	+	+	−	−
$\cos\theta$	+	−	−	+
$\tan\theta$	+	−	+	−

以下是一些例子

（4）$\theta = 30°$

$$\sin(90° + 30°) = \sin 120° = \cos 30° = \frac{\sqrt{3}}{2}$$

$$\sin(180° + 30°) = \sin 210° = -\sin 30° = -\frac{1}{2}$$

$$\sin(270° + 30°) = \sin 300° = -\cos 30° = -\frac{\sqrt{3}}{2}$$

（5）$\theta = 30°$

$$\cos(90° + 30°) = \cos 120° = -\sin 30° = -\frac{1}{2}$$

$$\cos(180° + 30°) = \cos 210° = -\cos 30° = -\frac{\sqrt{3}}{2}$$

$$\cos(270° + 30°) = \cos 300° = \sin 30° = \frac{1}{2}$$

（6）$\theta = 90°$

$$\sin(90° + 90°) = \sin 180° = \cos 90° = 0$$

$$\sin(180° + 90°) = \sin 270° = -\sin 90° = -1$$

$$\sin(270° + 90°) = \sin 360° = -\cos 90° = 0$$

（7）$\theta = 90°$

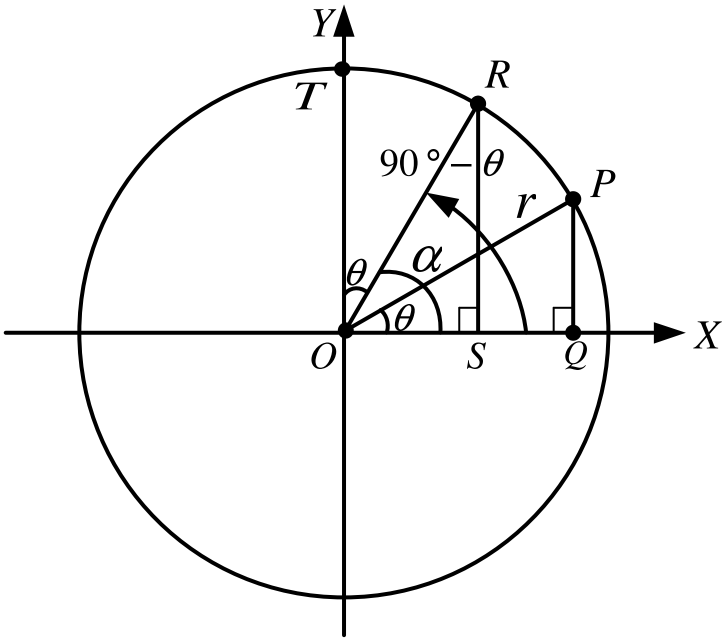
$$\cos(90° + 90°) = \cos 180° = -\sin 90° = -1$$

$$\cos(180° + 90°) = \cos 270° = -\cos 90° = 0$$

$$\cos(270° + 90°) = \cos 360° = \sin 90° = 1$$

以下我們要看另一種轉換

(8) $90° - \theta$



$\overline{PQ} = y$

$\overline{OQ} = x$

$\angle ROT = \theta$

$\therefore \quad \angle ROQ = 90° - \theta = \alpha$

我們可以證明 $\Delta ROS \equiv \Delta POQ$

$\overline{OR} = \overline{OQ} = x$

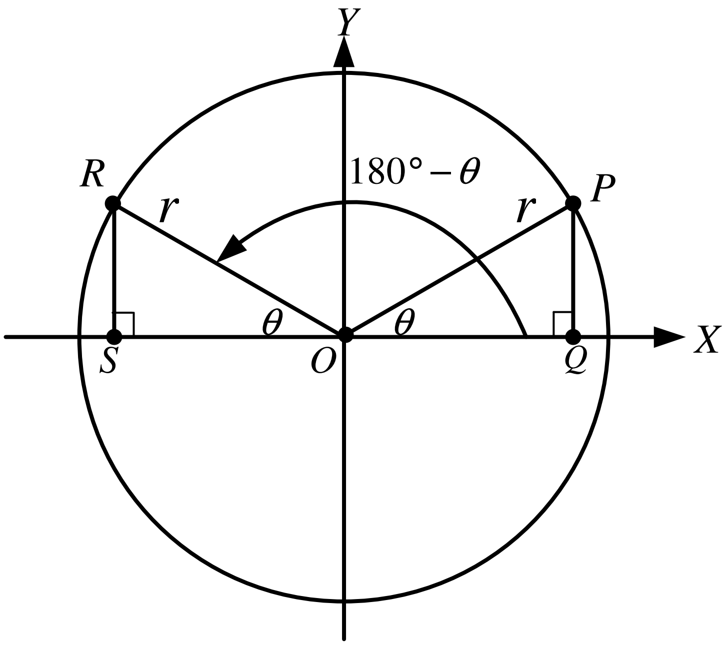
$\overline{RS} = \overline{PQ} = y$

$$\sin(90° - \theta) = \sin\alpha = \frac{\overline{RS}}{r} = \frac{x}{r} = \cos\theta$$

$$\cos(90° - \theta) = \cos\alpha = \frac{\overline{OS}}{r} = \frac{y}{x} = \sin\theta$$

$\tan(90° - \theta) = \tan\alpha = \frac{\overline{RS}}{\overline{OS}} = \frac{x}{y} = \cot\theta$

(9) 180° − θ



$\overline{PQ} = y$

$\overline{OQ} = x$

∠POR = 180° − θ

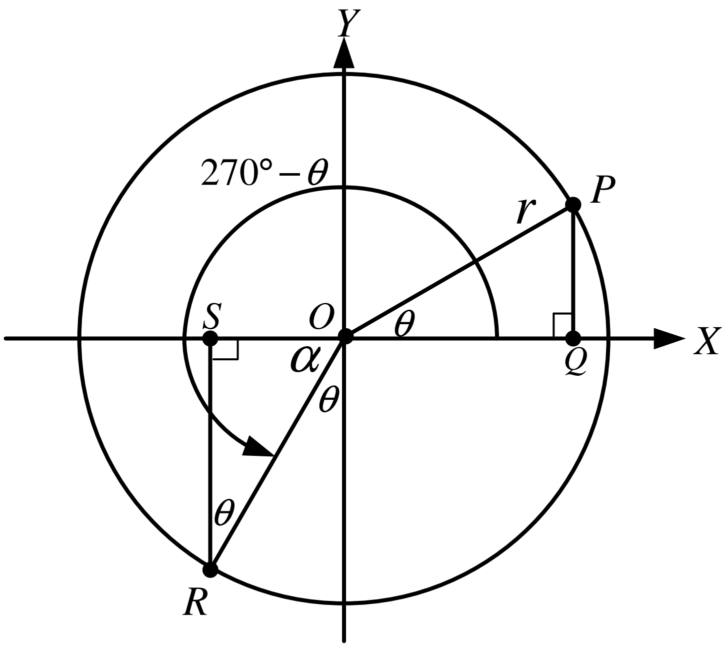
用同樣的方法，我們可以證明

$\sin(180° - \theta) = \frac{y}{r} = \sin\theta$

$\cos(180° - \theta) = \frac{-x}{r} = -\cos\theta$ (x在第三象限為負)

$\tan(180° - \theta) = \frac{y}{-x} = -\tan\theta$

(10) 270° − θ



$\overline{PQ} = y$

$\overline{OQ} = x$

$\angle \mathrm{QOR} = 180°$

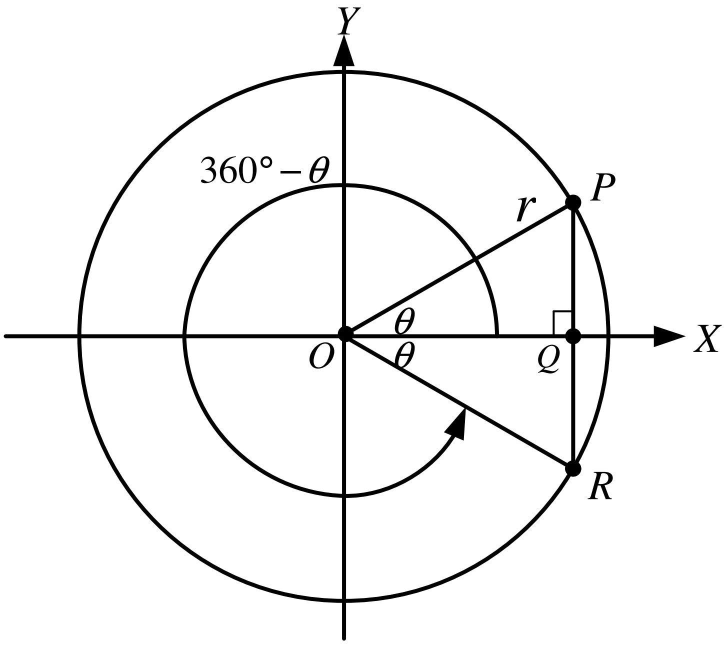
可以用同樣的方法證明

$\sin(270° - \theta) = \sin\alpha = \frac{-x}{r} = -\cos\theta$ （y在第三象限為負）

$\cos(270° - \theta) = \cos\alpha = \frac{-y}{r} = -\sin\theta$ （x在第三象限為負）

$\tan(270° - \theta) = \tan\alpha = \frac{y}{-x} = -\tan\theta$

（11）$360° - \theta$



$\overline{PQ} = y$

$\overline{OQ} = x$

$\angle \mathrm{POR} = 360° - \theta$

我們可以用同樣的方法證明

$\sin(360° - \theta) = \frac{-y}{r} = -\sin\theta$ （y在第四象限為負）

$\cos(360° - \theta) = \frac{x}{r} = \cos\theta$

$\tan(360° - \theta) = \frac{-y}{x} = -\tan\theta$

我們可以將以上的公式整理如下:

$\sin(90° - \theta) = \cos\theta$

$\sin(180° - \theta) = \sin\theta$

$\sin(270° - \theta) = -\cos\theta$

$\sin(360° - \theta) = -\sin\theta$

$\cos(90° - \theta) = \sin\theta$

$\cos(180° - \theta) = -\cos\theta$

$\cos(270° - \theta) = -\sin\theta$

$\cos(360° - \theta) = \cos\theta$

(12) $\theta = 30°$

$\sin(90° - 30°) = \sin 60° = \cos 30° = \frac{\sqrt{3}}{2}$

$\sin(180° - 30°) = \sin 150° = \sin 30° = \frac{1}{2}$

$\sin(270° - 30°) = \sin 240° = -\cos 30° = -\frac{\sqrt{3}}{2}$

$\sin(360° - 30°) = \sin 330° = -\sin 30° = -\frac{1}{2}$

(13) $\theta = 30°$

$\cos(90° - 30°) = \cos 60° = \sin 30° = \frac{1}{2}$

$\cos(180° - 30°) = \cos 150° = -\cos 30° = -\frac{\sqrt{3}}{2}$

$\cos(270° - 30°) = \cos 240° = -\sin 30° = -\frac{1}{2}$

$\cos(360° - 30°) = \cos 330° = \cos 30° = \frac{\sqrt{3}}{2}$

我們可以將以上所有的公式整理如下:

$\sin(90° + \theta) = \cos\theta \cdots\cdots (11.1)$

$\sin(90° - \theta) = \cos\theta \cdots\cdots (11.2)$

$\cos(90° + \theta) = -\sin\theta \cdots\cdots (11.3)$

$\cos(90° - \theta) = \sin\theta \cdots\cdots (11.4)$

$\sin(180° + \theta) = -\sin\theta \cdots\cdots (11.5)$

$\sin(180° - \theta) = \sin\theta \cdots\cdots (11.6)$

$\cos(180° + \theta) = -\cos\theta \cdots\cdots (11.7)$

$\cos(180° - \theta) = -\cos\theta \cdots\cdots (11.8)$

$\sin(270° + \theta) = -\cos\theta \cdots\cdots (11.9)$

$\sin(270° - \theta) = -\cos\theta \cdots\cdots (11.10)$
$\cos(270° + \theta) = \sin\theta \cdots\cdots (11.11)$

$\cos(270° - \theta) = -\sin\theta \cdots\cdots (11.12)$

$\sin(360° - \theta) = -\sin\theta \cdots\cdots (11.13)$

$\cos(360° - \theta) = \cos\theta \cdots\cdots (11.14)$

我們可以利用以上的公式發現很多有趣的現象，以下的例子很值得同學們看的:

(14)$120°$　$120° = 90° + 30° = 180° - 60°$

根據(11.1)

$\sin 120° = \sin(90° + 30°) = \cos 30° = \frac{\sqrt{3}}{2}$

根據(11.6)

$$\sin 120° = \sin(180° - 60°) = \sin 60° = \frac{\sqrt{3}}{2}$$

(15) $210°$ $210° = 180° + 30° = 270° - 60°$

根據(11.5)

$$\sin 210° = \sin(180° + 30°) = -\sin 30° = -\frac{1}{2}$$

根據(11.10)

$$\sin 210° = \sin(270° - 60°) = -\cos 60° = -\frac{1}{2}$$

(16) $300°$ $300° = 270° + 30° = 360° - 60°$

根據(11.9)

$$\sin 300° = \sin(270° + 30°) = -\cos 30° = -\frac{\sqrt{3}}{2}$$

根據(11.13)

$$\sin 300° = \sin(360° - 60°) = -\sin 60° = -\frac{\sqrt{3}}{2}$$